

COMPUTATION OF EXPANSIONS FOR THE MAXIMUM LIKELIHOOD ESTIMATOR AND ITS DISTRIBUTION FUNCTION

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Abstract. In this paper, insight is given in the techniques used to compute asymptotic expansions. In a broad fashion the technique is described. Most of the results apply to the paper "An expansion for the maximum likelihood estimator and its distribution function", which will be submitted.

Key words. asymptotic expansion, maximum likelihood estimator of location

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1. Expansions. In the paper "An expansion for the maximum likelihood estimator and its distribution function", which will be submitted, many expansions are being calculated. Because the technique itself is not so difficult and the outcomes take a lot of space to present, many steps of the computation of the expansions will be omitted in that paper. However if one wants to check the computations, it may be useful to have some insight in how the expansions were obtained. None of the applied techniques are claimed by the author. They are just written down. We will give the full outcome of all the steps needed to construct the needed expansions. We will not give any proofs, just the method of obtaining the expansions. This means that certain rest terms will be omitted. Also note that the results are only valid for the maximum likelihood estimator of location.

2. An expansion for the maximum likelihood estimator. We define the maximum likelihood estimator $\hat{\theta}_n$ by

$$(2.1) \quad L_n(\hat{\theta}_n) = \inf_{\theta \in R} L_n(\theta),$$

where $L_n(\theta) = n^{-1} \sum_{i=1}^n \rho(X_i - \theta)$, where $\rho(\cdot) = -\log(\cdot)$. It may be proved that (cf. Chibisov (1973)) $\hat{\theta}_n$ is the solution of the equation

$$(2.2) \quad L'_n(\theta) = 0,$$

with probability $1 + o(n^{-3/2})$. We expand $L'_n(\theta)$ with a Taylor expansion and get

$$(2.3) \quad \begin{aligned} L'_n(\theta) &= \frac{1}{n} \sum_{i=1}^n \rho'(X_i - \theta) = \frac{1}{n} \sum_{i=1}^n \rho'(X_i) - \frac{\theta}{n} \sum_{i=1}^n \rho''(X_i) + \frac{\theta^2}{2n} \sum_{i=1}^n \rho^{(3)}(X_i) \\ &\quad - \frac{\theta^3}{6n} \sum_{i=1}^n \rho^{(4)}(X_i) + \frac{\theta^4}{24n} \sum_{i=1}^n \rho^{(5)}(X_i - \theta'), \end{aligned}$$

where $|\theta'| \leq |\theta|$.

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We introduce the notation

$$(2.4) \quad \xi_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\rho^{(j)}(X_i) - a_j), \quad a_j = E_0 \rho^{(j)}(X) \quad \text{for } j = 1, \dots, 5.$$

Note that the ξ'_j s are normalized sums and that $a_1 = 0$. (2.3) becomes

$$(2.5) \quad \begin{aligned} L'_n(\theta) = & \frac{\xi_{1n}}{\sqrt{n}} - \theta \left(\frac{\xi_{2n}}{\sqrt{n}} + a_2 \right) + \frac{\theta^2}{2} \left(\frac{\xi_{3n}}{\sqrt{n}} + a_3 \right) \\ & - \frac{\theta^3}{6} \left(\frac{\xi_{4n}}{\sqrt{n}} + a_4 \right) + \frac{\theta^4}{24} \left(\frac{\xi_{5n}}{\sqrt{n}} + a_5 \right) + \dots, \end{aligned}$$

To find the expansion for $\hat{\theta}_n$ we put

$$(2.6) \quad \hat{\theta}_n = B_1/\sqrt{n} + B_2/n + B_3/n^{3/2} + B_4/n^2.$$

Substituting this into (2.5) leads to

$$(2.7) \quad \begin{aligned} & (-a_2 B_1 + \xi_1)/n^{1/2} + (-\xi_2 B_1 - a_2 B_2 + \frac{1}{2} a_3 B_1^2)/n \\ & + (-\xi_2 B_2 - \frac{1}{6} a_4 B_1^3 + \frac{1}{2} \xi_3 B_1^2 + a_3 B_1 B_2 - a_2 B_3)/n^{3/2} \\ & + (-\frac{1}{2} a_4 B_1^2 B_2 + \frac{1}{2} a_3 B_2^2 + \frac{1}{24} a_5 B_1^4 - \xi_2 B_3 + \xi_3 B_1 B_2 - \frac{1}{6} \xi_4 B_1^3 + a_3 B_1 B_3 - a_2 B_4)/n^2 \end{aligned}$$

Now we take the first term of (2.7) and put it equal to 0. We get

$$(2.8) \quad -a_2 B_1 + \xi_1 = 0 \Rightarrow B_1 = \xi_1/a_2.$$

Substituting the obtained B_1 in (2.7) gives

$$(2.9) \quad \begin{aligned} & (-\xi_2 \xi_1/a_2 + \frac{1}{2} a_3 \xi_1^2/a_2^2 - a_2 B_2)/n \\ & + (-a_2 B_3 + a_3 \xi_1 B_2/a_2 - \xi_2 B_2 + \frac{1}{2} \xi_3 \xi_1^2/a_2^2 - \frac{1}{6} a_4 \xi_1^3/a_2^3)/n^{3/2} \\ & + (a_3 \xi_1 B_3/a_2 + \frac{1}{24} a_5 \xi_1^4/a_2^4 - \frac{1}{2} a_4 \xi_1^2 B_2/a_2^2 - \\ & \quad \frac{1}{6} \xi_4 \xi_1^3/a_2^3 + \frac{1}{2} a_3 B_2^2 + \xi_3 \xi_1 B_2/a_2 - \xi_2 B_3 - a_2 B_4)/n^2 \end{aligned}$$

Note that the $1/\sqrt{n}$ term has vanished. We take the first term of (2.9) and put it equal to 0.

This results in

$$(2.10) \quad B_2 = -\xi_1 \xi_2/a_2^2 + \frac{1}{2} \xi_1^2 a_3/a_2^3.$$

We substitute this B_2 in (2.9) and get

$$(2.11) \quad \begin{aligned} & (-a_2 B_3 - \frac{1}{6} a_4 \xi_1^3/a_2^3 + \frac{1}{2} \xi_3 \xi_1^2/a_2^2 - \frac{3}{2} \xi_2 \xi_1^2 a_3/a_2^3 + \xi_2^2 \xi_1/a_2^2 + \frac{1}{2} \xi_1^3 a_3^2/a_2^4)/n^{3/2} \\ & + (-\frac{1}{6} \xi_4 \xi_1^3/a_2^3 + a_3 \xi_1 B_3/a_2 + \frac{1}{8} \xi_1^4 a_3^3/a_2^6 + \frac{1}{2} \xi_3 \xi_1^3 a_3/a_2^4 \\ & - \frac{1}{2} \xi_1^3 \xi_2 a_2^2/a_2^5 - \xi_2 B_3 - \xi_3 \xi_1^2 \xi_2/a_2^3 \\ & + \frac{1}{24} a_5 \xi_1^4/a_2^4 + \frac{1}{2} a_3 \xi_1^2 \xi_2^2/a_2^4 + \frac{1}{2} a_4 \xi_1^3 \xi_2/a_2^4 - a_2 B_4 - \frac{1}{4} a_4 \xi_1^4 a_3/a_2^5)/n^2 \end{aligned}$$

Again we put the first term of the result to 0 and get

$$(2.12) \quad B_3 = \xi_1 \xi_2^2 / a_2^3 - \frac{1}{6} \xi_1^3 a_4 / a_2^4 + \frac{1}{2} \xi_1^2 \xi_3 / a_2^3 - \frac{3}{2} \xi_1^2 \xi_2 a_3 / a_2^4 + \frac{1}{2} \xi_1^3 a_3^2 / a_2^5.$$

We substitute B_3 in (2.11) to obtain

$$(2.13) \quad \begin{aligned} & -a_2 B_4 - \frac{5}{2} \xi_1^3 \xi_2 a_3^2 / a_2^5 + \frac{5}{8} \xi_1^4 a_3^3 / a_2^6 - \xi_2^3 \xi_1 / a_2^3 + \xi_3 \xi_1^3 a_3 / a_2^4 - \frac{3}{2} \xi_3 \xi_1^2 \xi_2 / a_2^3 \\ & + \frac{2}{3} a_4 \xi_1^3 \xi_2 / a_2^4 + \frac{1}{24} a_5 \xi_1^4 / a_2^4 - \frac{1}{6} \xi_4 \xi_1^3 / a_2^3 + 3 a_3 \xi_1^2 \xi_2^2 / a_2^4 - \frac{5}{12} a_4 \xi_1^4 a_3 / a_2^5. \end{aligned}$$

This is put equal to 0 and at last we obtain

$$(2.14) \quad \begin{aligned} B_4 = & \xi_1^3 \xi_3 a_3 / a_2^5 - \frac{5}{2} \xi_1^3 \xi_2 a_3^2 / a_2^6 + \frac{5}{8} \xi_1^4 a_3^3 / a_2^7 - \xi_1 \xi_2^3 / a_2^4 - \frac{1}{6} \xi_1^3 \xi_4 / a_2^4 \\ & - \frac{3}{2} \xi_1^2 \xi_3 \xi_2 / a_2^4 + \frac{2}{3} \xi_1^3 a_4 \xi_2 / a_2^5 + \frac{1}{24} \xi_1^4 a_5 / a_2^5 + 3 \xi_1^2 a_3 \xi_2^2 / a_2^5 - \frac{5}{12} \xi_1^4 a_4 a_3 / a_2^6. \end{aligned}$$

Eventually

$$(2.15) \quad \begin{aligned} \sqrt{n}(\hat{\theta}_n) = & \frac{\xi_1}{a_2} + \frac{1}{\sqrt{n}} \left(\frac{-\xi_1 \xi_2}{a_2^2} + \frac{a_3 \xi_1^2}{2 a_2^3} \right) + \frac{1}{n} \left(\frac{\xi_1 \xi_2^2}{a_2^3} - \frac{3 a_3 \xi_1^2 \xi_2}{2 a_2^4} + \frac{\xi_1^2 \xi_3}{2 a_2^3} + \frac{a_3^2 \xi_1^3}{2 a_2^5} - \frac{a_4 \xi_1^3}{6 a_2^4} \right) \\ & + \frac{1}{n^{3/2}} \left(\frac{3 \xi_1^2 a_3 \xi_2^2}{a_2^5} + \frac{5 \xi_1^4 a_3^3}{8 a_2^7} - \frac{5 \xi_1^4 a_4 a_3}{12 a_2^6} - \frac{3 \xi_1^2 \xi_3 \xi_2}{2 a_2^4} - \frac{5 \xi_1^3 a_3^2 \xi_2}{2 a_2^6} + \frac{\xi_1^4 a_5}{24 a_2^5} \right. \\ & \left. + \frac{\xi_1^3 \xi_3 a_3}{a_2^5} + \frac{2 \xi_1^3 a_4 \xi_2}{3 a_2^5} - \frac{\xi_1^3 \xi_4}{6 a_2^4} - \frac{\xi_1 \xi_2^3}{a_2^4} \right) + \dots, \end{aligned}$$

3. Expansion for the distribution function of the maximum likelihood estimator. The estimator in (2.15) fits the model of Hall (1992), Section 2.3. This means that the cumulants of $\sqrt{n}\hat{\theta}_n$ will determine the expansion for the distribution function. First we note that the cumulants will be of the form (cf. Hall(1992), Section 2.3)

$$(3.1) \quad \begin{aligned} \kappa_1 &= 0 + k_{12}/\sqrt{n} + k_{13}/n^{3/2} + \dots \\ \kappa_2 &= k_{21} + k_{22}/n + \dots \\ \kappa_3 &= k_{31}/\sqrt{n} + k_{32}/n^{3/2} + \dots \\ \kappa_4 &= k_{41}/n + \dots \\ \kappa_5 &= k_{51}/n^{3/2} + \dots \end{aligned}$$

Furthermore (cf. Hall(1992), Section 2.2)

$$(3.2) \quad \begin{aligned} \kappa_1 &= ES_n \\ \kappa_2 &= ES_n^2 - (ES_n)^2 \\ \kappa_3 &= E(S_n - ES_n)^3 = ES_n^3 - 3ES_n^2 ES_n + 2(ES_n)^3 \\ \kappa_4 &= E(S_n - ES_n)^4 - 3\kappa_2^2 \\ \kappa_5 &= E(S_n - ES_n)^5 - 10\kappa_2 \kappa_3. \end{aligned}$$

3.1. Computation of the kappa's. We will now compute the kappa's. For the computation of the kappa's we need to calculate the expectation of the normalized sums, the so called ξ_j 's. Terms that become too small will be omitted. First we will introduce the notation

$$(3.3) \quad \psi_i(\cdot) = \frac{f^{(i)}}{f}(\cdot)$$

$$(3.4) \quad \begin{aligned} \eta_2 &= E(\psi_2^2(X_i)), & \eta_3 &= E(\psi_1^3(X_i)), & \eta_4 &= E(\psi_1^4(X_i)), \\ \eta_5 &= E(\psi_1^5(X_i)), & \eta_6 &= E(\psi_2\psi_3(X_i)) \end{aligned}$$

the above results in

$$(3.5) \quad \begin{aligned} a_1 &= 0, & a_2 &= E(\psi_1^2(X_i)), \text{ without loss of generality we put } a_2 = 1, \\ a_3 &= -\frac{1}{2}\eta_3, & a_4 &= \frac{2}{3}\eta_4 - \eta_2, & a_5 &= 5\eta_6 - \frac{3}{2}\eta_5. \end{aligned}$$

and that

$$E(\psi_1\psi_2) = \frac{1}{2}\eta_3, \quad E(\psi_1\psi_3) = -\eta_2 + \frac{2}{3}\eta_4, \quad E(\psi_1\psi_4) = -5\eta_6 + \frac{3}{2}\eta_5$$

$$E(\psi_1^2\psi_2) = \frac{2}{3}\eta_4, \quad E(\psi_1\psi_2^2) = 2\eta_6, \quad E(\psi_1^3\psi_2) = \frac{3}{4}\eta_5, \quad E(\psi_1^2\psi_3) = -4\eta_6 + \frac{3}{2}\eta_5$$

Furthermore

$$w_j = -(\rho^{(j)}(X_i) - a_j), \text{ for } j = 1, \dots, 5.$$

Consequently,

$$(3.6) \quad \begin{aligned} w_1 &= \psi_1, \\ w_2 &= \psi_2 - \psi_1^2 + 1, \\ w_3 &= \psi_3 - 3\psi_1\psi_2 + 2\psi_1^3 + a_3, \\ w_4 &= \psi_4 - 4\psi_1\psi_3 + 12\psi_1^2\psi_2 - 3\psi_2^2 - 6\psi_1^4 + a_4 \\ w_5 &= \psi_5 - 5\psi_1\psi_4 + 20\psi_1^2\psi_3 - 10\psi_2\psi_3 + 30\psi_1\psi_2^2 - 60\psi_1^3\psi_2 + 24\psi_1^5 + a_5 \end{aligned}$$

Note that $Ew_j = 0$ for $j = 1, \dots, 5$ and that $E(w_1^2) = 1$.

$$E(w_1w_2) = -\frac{1}{2}\eta_3, \quad E(w_1w_3) = \frac{2}{3}\eta_4 - \eta_2, \quad E(w_1w_4) = 5\eta_6 - \frac{3}{2}\eta_5$$

$$E(w_1^2w_2) = -\frac{1}{3}\eta_4 + 1, \quad E(w_1^2w_3) = -4\eta_6 + \frac{5}{4}\eta_5 - \frac{1}{2}\eta_3$$

$$E(w_1^3w_2) = -\frac{1}{4}\eta_5 + \eta_3, \quad E(w_1w_2^2) = 2\eta_6 - \frac{1}{2}\eta_5 - \eta_3$$

$$E(w_2^2) = \eta_2 - \frac{1}{3}\eta_4 - 1, \quad E(w_2 w_3) = -\eta_6 + \frac{1}{4}\eta_5 + \frac{1}{2}\eta_3$$

We will give an example here to illustrate how the expectations of the terms of $\sqrt{n}\hat{\theta}_n$ are obtained.

$$\begin{aligned}
 (3.7) \quad E\xi_1^2 &= E\left(\frac{1}{\sqrt{n}} \sum_i w_{1i}\right)^2 \\
 &= E\left\{\frac{1}{n} \left(\sum_i w_{1i}^2 + 2 \sum_{i < j} w_{1i} w_{1j}\right)\right\} \\
 &= \frac{1}{n} (n E w_1^2 + \frac{2n(n-1)}{2} E w_1 E w_1) = E w_1^2 + 0 = E w_1^2 = 1.
 \end{aligned}$$

$$\begin{aligned}
 (3.8) \quad E\xi_1^8 &= E\left(\frac{1}{\sqrt{n}} \sum_i w_{1i}\right)^8 \\
 &= \frac{1}{n^4} E \left[\sum_i w_{1i}^8 + 8 \sum_{i \neq j} w_{1i}^7 w_{1j} + \binom{8}{2} \sum_{i \neq j} w_{1i}^6 w_{1j}^2 \right. \\
 &\quad + \binom{8}{3} \sum_{i \neq j} w_{1i}^5 w_{1j}^3 + \binom{8}{4} \sum_{i < j} w_{1i}^4 w_{1j}^4 \\
 &\quad + \binom{8}{4, 2, 2} \sum \sum \sum_{i \neq j < k \neq i} w_{1i}^4 w_{1j}^2 w_{1k}^2 + \binom{8}{2, 3, 3} \sum \sum \sum_{i \neq j < k \neq i} w_{1i}^2 w_{1j}^3 w_{1k}^3 \\
 &\quad \left. + \binom{8}{2, 2, 2, 2} \sum \sum \sum \sum_{i < j < k < l} w_{1i}^2 w_{1j}^2 w_{1k}^2 w_{1l}^2 \right] \\
 &= \frac{1}{n^4} \left[\binom{8}{2, 2, 2, 2} \frac{n(n-1)(n-2)(n-3)}{4!} (E w_1^2)^4 \right. \\
 &\quad + \binom{8}{2, 2, 4} \frac{n(n-1)(n-2)}{2!} E w_1^4 (E w_1^2)^2 \\
 &\quad + \left. \binom{8}{2, 3, 3} \frac{n(n-1)(n-2)}{2!} (E w_1^3)^2 E w_1^2 + \dots \right] \\
 &= \frac{1}{n^4} \left[105(n^4 - 6n^3 + 11n^2 - 6n)(E w_1^2)^4 + 210(n^3 - 3n^2 + 2n) E w_1^4 (E w_1^2)^2 \right. \\
 &\quad \left. + 280(n^3 - 3n^2 + 2n)(E w_1^3)^2 E w_1^2 + \dots \right] \\
 &= 105(E w_1^2)^4 + \frac{1}{n} [-630(E w_1^2)^4 + 210 E w_1^4 (E w_1^2)^2 + 280(E w_1^3)^2 E w_1^2] + \dots \\
 &= 105 + \frac{1}{n} [-630 + 210\eta_4 + 280\eta_3^2] + \dots
 \end{aligned}$$

The following equations give formula's for the other expectation that have to be computed.

$$\begin{aligned}
(3.9) \quad E(\xi_1^{2k}) &= \binom{2k}{2, \dots, 2} \frac{1}{k!} (E(w_1^2))^k + \frac{1}{n} \left[-\binom{2k}{2, \dots, 2} \frac{\binom{k}{2}}{k!} (Ew_1^2)^k \right. \\
&\quad + \binom{2k}{4, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^4 (Ew_1^2)^{k-2} \\
&\quad \left. + \binom{2k}{3, 3, 2, \dots, 2} \frac{1}{2} \frac{1}{(k-3)!} (Ew_1^3)^2 (Ew_1^2)^{k-3} \right] + \dots
\end{aligned}$$

$$\begin{aligned}
(3.10) \quad E(\xi_1^{2k+1}) &= -\frac{1}{\sqrt{n}} \left[\binom{2k+1}{3, 2, \dots, 2} \frac{1}{(k-1)!} (Ew_1^2)^{k-1} Ew_1^3 \right] \\
&\quad - \frac{1}{n\sqrt{n}} \left[-\binom{k}{2} \binom{2k+1}{3, 2, \dots, 2} \frac{1}{(k-1)!} (Ew_1^2)^{k-1} Ew_1^3 \right. \\
&\quad + \binom{2k+1}{5, 2, \dots, 2} \frac{1}{(k-2)!} (Ew_1^2)^{k-2} Ew_1^5 \\
&\quad \left. + \binom{2k+1}{4, 3, 2, \dots, 2} \frac{1}{(k-3)!} (Ew_1^2)^{k-3} Ew_1^4 Ew_1^3 \right] + \dots
\end{aligned}$$

For $j = 2, \dots, 4$ we have

$$\begin{aligned}
(3.11) \quad E(\xi_1^{2k} \xi_j) &= -\frac{1}{\sqrt{n}} \left[\binom{2k}{2, \dots, 2} \frac{1}{(k-1)!} Ew_1^2 w_j (Ew_1^2)^{k-1} \right. \\
&\quad + \binom{2k}{1, 3, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^3 Ew_1 w_j (Ew_1^2)^{k-2} \left. \right] \\
&\quad - \frac{1}{n\sqrt{n}} \left[-\frac{\binom{k}{2}}{(k-1)!} \binom{2k}{2, \dots, 2} Ew_1^2 w_j (Ew_1^2)^{k-1} \right. \\
&\quad - \frac{\binom{k}{2}}{(k-2)!} \binom{2k}{1, 3, 2, \dots, 2} Ew_1^3 Ew_1 w_j (Ew_1^2)^{k-2} \\
&\quad + \binom{2k}{5, 1, 2, \dots, 2} \frac{1}{(k-3)!} Ew_1^5 Ew_1 w_j (Ew_1^2)^{k-3} \\
&\quad + \binom{2k}{4, 2, \dots, 2} \frac{1}{(k-3)!} Ew_1^4 Ew_1^2 w_j (Ew_1^2)^{k-3} \\
&\quad + \binom{2k}{3, 3, 2, \dots, 2} \frac{1}{(k-3)!} Ew_1^3 Ew_1^3 w_j (Ew_1^2)^{k-3} \\
&\quad \left. + \binom{2k}{4, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^4 w_j (Ew_1^2)^{k-2} \right] + \dots
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad E(\xi_1^{2k+1} \xi_2) &= (2k+1)(2k-1) \dots 1 (E(w_1^2))^k Ew_1 w_2 \\
&\quad + \frac{1}{n} \left[-\frac{\binom{k+1}{2}}{k!} \binom{2k+1}{1, 2, \dots, 2} Ew_1 w_2 (Ew_1^2)^k \right. \\
&\quad \left. + \frac{1}{(k-2)!} \binom{2k+1}{4, 1, \dots, 2} Ew_1^4 Ew_1 w_2 (Ew_1^2)^{k-2} \right]
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \frac{1}{(k-3)!} \binom{2k+1}{3, 3, 1, 2, \dots, 2} (Ew_1^3)^2 Ew_1 w_2 (Ew_1^2)^{k-3} \\
 & + \frac{1}{(k-1)!} \binom{2k+1}{3, 2, \dots, 2} Ew_1^3 w_2 (Ew_1^2)^{k-1} \\
 & + \frac{1}{(k-2)!} \binom{2k+1}{3, 2, \dots, 2} Ew_1^3 Ew_1^2 w_2 (Ew_1^2)^{k-2} \Big] + \dots
 \end{aligned}$$

$$\begin{aligned}
 (3.13) \quad E(\xi_1^{2k} \xi_2^2) &= \binom{2k}{2, \dots, 2} \frac{1}{k!} (Ew_1^2)^k Ew_2^2 \\
 &+ \binom{2k}{1, 1, 2, \dots, 2} \frac{1}{(k-1)!} (Ew_1^2)^{k-1} (Ew_1 w_2)^2 \\
 &+ \frac{1}{n} \Big[\binom{2k}{4, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^4 (Ew_1^2)^{k-2} Ew_2^2 \\
 &+ \binom{2k}{3, 3, 2, \dots, 2} \frac{1}{2 \cdot (k-3)!} (Ew_1^3)^2 (Ew_1^2)^{k-3} Ew_2^2 \\
 &+ \binom{2k}{4, 1, 1, 2, \dots, 2} \frac{1}{(k-3)!} Ew_1^4 (Ew_1 w_2)^2 (Ew_1^2)^{k-3} \\
 &+ \binom{2k}{1, 3, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^3 Ew_1 w_2^2 (Ew_1^2)^{k-2} \\
 &+ \binom{2k}{1, 3, 2, \dots, 2} \frac{\binom{2}{1}}{(k-2)!} Ew_1^3 Ew_1^2 w_2 Ew_1 w_2 (Ew_1^2)^{k-2} \\
 &+ \binom{2k}{1, 3, 2, \dots, 2} \frac{\binom{2}{1}}{(k-2)!} Ew_1^3 w_2 Ew_1 w_2 (Ew_1^2)^{k-2} \\
 &- \binom{2k}{2, \dots, 2} \frac{\binom{k+1}{2}}{k!} (Ew_1^2)^k Ew_2^2 \\
 &+ \binom{2k}{2, \dots, 2} \frac{\binom{2}{1}}{2(k-2)!} (Ew_1^2 w_2)^2 (Ew_1^2)^{k-2} \\
 &- \binom{2k}{1, 1, 2, \dots, 2} \frac{\binom{k+1}{2}}{(k-1)!} (Ew_1^2)^{k-1} (Ew_1 w_2)^2 \\
 &+ \binom{2k}{2, \dots, 2} \frac{1}{(k-1)!} (Ew_1^2 w_2^2 (Ew_1^2)^{k-1}) \Big] + \dots
 \end{aligned}$$

$$\begin{aligned}
 (3.14) \quad E(\xi_1^{2k} \xi_2 \xi_3) &= \binom{2k}{2, \dots, 2} \frac{1}{k!} (Ew_1^2)^k Ew_2 w_3 \\
 &+ \frac{1}{(k-1)!} \binom{2k}{1, 1, 2, \dots, 2} Ew_1 w_2 Ew_1 w_3 + \dots
 \end{aligned}$$

$$(3.15) \quad E(\xi_1^{2k+1} \xi_2 \xi_3) = -\frac{1}{\sqrt{n}} \Big[\binom{2k+1}{3, 2, \dots, 2} \frac{1}{(k-1)!} Ew_1^3 Ew_2 w_3 (Ew_1^2)^{k-1}$$

$$\begin{aligned}
& + \binom{2k+1}{1, 1, 3, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^3 Ew_1 w_2 Ew_1 w_3 (Ew_1^2)^{k-2} \\
& + \binom{2k+1}{1, 2, \dots, 2} \frac{1}{(k-1)!} Ew_1^2 w_2 Ew_1 w_3 (Ew_1^2)^{k-1} \\
& + \binom{2k+1}{1, 2, \dots, 2} \frac{1}{(k-1)!} Ew_1 w_2 Ew_1^2 w_3 (Ew_1^2)^{k-1} \\
& + \binom{2k+1}{1, 2, \dots, 2} \frac{1}{k!} Ew_1 w_2 w_3 (Ew_1^2)^k \Big] + \dots
\end{aligned}$$

$$\begin{aligned}
(3.16) \quad E(\xi_1^{2k+1} \xi_2^2) &= -\frac{1}{\sqrt{n}} \Big[\binom{2k+1}{3, 2, \dots, 2} \frac{1}{(k-1)!} Ew_1^3 Ew_2^2 (Ew_1^2)^{k-1} \\
& + \binom{2k+1}{1, 1, 3, 2, \dots, 2} \frac{1}{(k-2)!} Ew_1^3 (Ew_1 w_2)^2 (Ew_1^2)^{k-2} \\
& + \binom{2k+1}{1, 2, \dots, 2} \frac{1}{k!} Ew_1 w_2^2 (Ew_1^2)^k \\
& + \binom{2k+1}{1, 2, \dots, 2} \frac{\binom{2}{1}}{(k-1)!} Ew_1^2 w_2 Ew_1 w_2 (Ew_1^2)^{k-1} \Big] + \dots
\end{aligned}$$

$$\begin{aligned}
(3.17) \quad E(\xi_1^{2k+1} \xi_2^3) &= \binom{2k+1}{1, 2, \dots, 2} \frac{\binom{3}{1}}{k!} Ew_1 w_2 Ew_2^2 (Ew_1^2)^k \\
& + \binom{2k+1}{1, 1, 1, 2, \dots, 2} \binom{3}{111} \frac{1}{3!(k-1)!} (Ew_1 w_2)^3 (Ew_1^2)^{k-1} + \dots
\end{aligned}$$

$$\begin{aligned}
(3.18) \quad E(\xi_1^{2k} \xi_2^3) &= -\frac{1}{\sqrt{n}} \Big[\binom{2k}{1, 3, 2, \dots, 2} \frac{\binom{3}{1}}{(k-2)!} Ew_1^3 Ew_2^2 Ew_1 w_2 (Ew_1^2)^{k-2} \\
& + \frac{1}{k!} \binom{2k}{2, \dots, 2} Ew_2^3 (Ew_1^2)^k \\
& + \binom{2k}{2, \dots, 2} \frac{3}{(k-1)!} Ew_1^2 w_2 Ew_2^2 (Ew_1^2)^{k-1} \\
& + \frac{1}{(k-3)!} \cdot \binom{2k}{1, 1, 1, 3, 2, \dots, 2} Ew_1^3 (Ew_1 w_2)^3 (Ew_1^2)^{k-3} \\
& + \binom{2k}{1, 1, 2, \dots, 2} \frac{3}{(k-1)!} (Ew_1^2)^{k-1} Ew_1 w_2^2 Ew_1 w_2 \\
& + \binom{2k}{1, 1, 2, \dots, 2} \frac{3}{(k-2)!} (Ew_1^2)^{k-2} Ew_1^2 w_2 (Ew_1 w_2)^2 \Big] + \dots
\end{aligned}$$

$$(3.19) \quad E(\xi_1^{2k} \xi_2^4) = \binom{2k}{2, \dots, 2} \frac{3}{k!} (Ew_2^2)^2 (Ew_1^2)^k$$

$$\begin{aligned}
 & + \binom{2k}{1, 1, 2, \dots, 2} \frac{6}{(k-1)!} (Ew_1^2)^{k-1} Ew_2^2 (Ew_1 w_2)^2 \\
 & + \binom{2k}{1, 1, 1, 1, 2, \dots, 2} \frac{1}{(k-2)!} (Ew_1^2)^{k-2} (Ew_1 w_2)^4 + \dots
 \end{aligned}$$

Further computations lead to

$$\begin{aligned}
 (3.20) \quad ES_n &= \frac{\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left(\frac{1}{9} \eta_4 \eta_3 + \frac{1}{16} \eta_5 - \frac{1}{4} \eta_3 + \frac{5}{24} \eta_3 \eta_2 - \frac{11}{64} \eta_3^3 - \frac{3}{8} \eta_6 \right) + \dots \\
 ES_n^2 &= 1 + \frac{1}{n} \left(-\frac{1}{16} \eta_3^2 - \frac{1}{3} \eta_4 + \eta_2 - 1 \right) + \dots \\
 ES_n^3 &= \frac{5\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left(-\frac{5}{12} \eta_4 \eta_3 + \frac{35}{8} \eta_3 \eta_2 - \frac{45}{32} \eta_3^3 - \frac{15}{4} \eta_3 - \frac{45}{8} \eta_6 + \frac{21}{16} \eta_5 \right) + \dots \\
 ES_n^4 &= 3 + \frac{1}{n} \left(10\eta_2 - 9 + \frac{1}{8} \eta_3^2 - \frac{11}{3} \eta_4 \right) + \dots \\
 ES_n^5 &= \frac{35\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left(-\frac{175}{12} \eta_4 \eta_3 - \frac{525}{8} \eta_6 - \frac{875}{64} \eta_3^3 + \frac{259}{16} \eta_5 + \frac{525}{8} \eta_3 \eta_2 - \frac{105}{2} \eta_3 \right) \\
 & + \dots
 \end{aligned}$$

Finally, by using (3.2) we get the cumulants,

$$\begin{aligned}
 (3.21) \quad \kappa_1 &= \frac{\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left(\frac{1}{9} \eta_4 \eta_3 + \frac{1}{16} \eta_5 - \frac{1}{4} \eta_3 + \frac{5}{24} \eta_3 \eta_2 - \frac{11}{64} \eta_3^3 - \frac{3}{8} \eta_6 \right) + \dots \\
 \kappa_2 &= 1 + \frac{1}{n} \left(-\frac{1}{8} \eta_3^2 - 1 - \frac{1}{3} \eta_4 + \eta_2 \right) + \dots \\
 \kappa_3 &= \frac{\eta_3}{2\sqrt{n}} + \frac{1}{n^{3/2}} \left(-\frac{9}{4} \eta_3 + 3\eta_3 \eta_2 - \frac{1}{2} \eta_3 \eta_4 + \frac{9}{8} \eta_5 - \frac{13}{16} \eta_3^3 - \frac{9}{2} \eta_6 \right) + \dots \\
 \kappa_4 &= \frac{1}{n} \left(-3 - \frac{5}{3} \eta_4 + 4\eta_2 \right) + \dots \\
 \kappa_5 &= \frac{1}{n^{3/2}} \left(-15\eta_6 - 10\eta_3 - 5\eta_3 \eta_4 + 15\eta_3 \eta_2 + 4\eta_5 - \frac{15}{8} \eta_3^3 \right) + \dots
 \end{aligned}$$

Note that indeed they are of the form in (3.1).

3.2. Finding the polynomials of the expansion. We will now find the polynomials which make up the expansion of the distribution function.

We view the characteristic function as

$$(3.22) \quad \exp\left\{ \kappa_1(it) + \frac{1}{2} \kappa_2(it)^2 + \frac{1}{6} \kappa_3(it)^3 + \frac{1}{24} \kappa_4(it)^4 + \frac{1}{120} \kappa_5(it)^5 + \dots \right\}$$

By using (3.1) we get

$$\exp\left\{ (it) \left(\frac{k_{12}}{\sqrt{n}} + \frac{k_{13}}{n\sqrt{n}} \right) + \frac{1}{2} (it)^2 \left(1 + \frac{k_{22}}{n} \right) + \frac{1}{6} (it)^3 \left(\frac{k_{31}}{\sqrt{n}} + \frac{k_{32}}{n\sqrt{n}} \right) \right\}$$

$$\begin{aligned}
& + \frac{1}{24}(it)^4 \frac{k_{41}}{n} + \frac{1}{120}(it)^5 \frac{k_{51}}{n\sqrt{n}} \} \\
& = \exp\{-\frac{1}{2}t^2\} \exp\{\frac{1}{\sqrt{n}}(k_{12}(it) + \frac{1}{6}k_{31}(it)^3) + \frac{1}{n}(\frac{1}{2}k_{22}(it)^2 \\
& + \frac{1}{24}k_{41}(it)^4) + \frac{1}{n^{3/2}}(k_{13}(it) + \frac{1}{6}k_{32}(it)^3 + \frac{1}{120}k_{51}(it)^5)\}
\end{aligned}$$

Constructing a Taylor expansion for the above gives

$$\begin{aligned}
(3.23) \quad & \exp\{-\frac{1}{2}t^2\} \exp\{\frac{1}{\sqrt{n}}(k_{12}(it) + \frac{1}{6}k_{31}(it)^3) + \frac{1}{n}(\frac{1}{2}k_{22}(it)^2 \\
& + \frac{1}{24}k_{41}(it)^4) + \frac{1}{n^{3/2}}(k_{13}(it) + \frac{1}{6}k_{32}(it)^3 + \frac{1}{120}k_{51}(it)^5)\} \\
& = \exp\{-\frac{1}{2}t^2\} \left[1 + \frac{1}{\sqrt{n}}(k_{12}(it) + \frac{1}{6}k_{31}(it)^3) + \frac{1}{n}(\frac{1}{2}k_{22}(it)^2 \right. \\
& + \frac{1}{24}k_{41}(it)^4) + \frac{1}{n^{3/2}}(k_{13}(it) + \frac{1}{6}k_{32}(it)^3 + \frac{1}{120}k_{51}(it)^5) \\
& + \frac{1}{2}(\frac{1}{n}(k_{12}(it) + \frac{1}{6}k_{31}(it)^3))^2 \\
& + \frac{2}{n^{3/2}}(k_{12}(it) + \frac{1}{6}k_{31}(it)^3)(\frac{1}{2}k_{22}(it)^2 + \frac{1}{24}k_{41}(it)^4 + \dots) \\
& \left. + \frac{1}{6}(\frac{1}{n^{3/2}}(k_{12}(it) + \frac{1}{6}k_{31}(it)^3))^3 \right] \\
& = \exp\{-\frac{1}{2}t^2\} (1 + \frac{r_1(it)}{\sqrt{n}} + \frac{r_2(it)}{n} + \frac{r_3(it)}{n\sqrt{n}}).
\end{aligned}$$

Consequently,

$$(3.24) \quad r_1(it) = ((it)k_{12} + \frac{1}{6}(it)^3k_{31})$$

$$(3.25) \quad r_2(it) = (\frac{1}{2}(it)^2k_{12}^2 + \frac{1}{72}(it)^6k_{31}^2 + \frac{1}{6}(it)^4k_{12}k_{31} + \frac{1}{24}(it)^4k_{41} + \frac{1}{2}(it)^2k_{22})$$

$$\begin{aligned}
(3.26) \quad r_3(it) = & (\frac{1}{72}(it)^7k_{12}k_{31}^2 + \frac{1}{24}(it)^5k_{12}k_{41} + \frac{1}{12}(it)^5k_{22}k_{31} + \frac{1}{144}(it)^7k_{31}k_{41} \\
& + \frac{1}{120}k_{51}(it)^5 + \frac{1}{12}(it)^5k_{12}^2k_{31} + \frac{1}{2}(it)^3k_{12}k_{22} \\
& + \frac{1}{6}(it)^3k_{32} + (it)k_{13} + \frac{1}{6}(it)^3k_{12}^3 + \frac{1}{1296}(it)^9k_{31}^3)
\end{aligned}$$

The polynomials of the expansions are now(cf. Hall(1992), Section 2.2)

$$(3.27) \quad p_1 = -\frac{1}{12}\eta_3(x^2 + 2)$$

$$(3.28) \quad p_2 = -\frac{1}{288}\eta_3^2x^5 + (\frac{1}{72}\eta_3^2 + \frac{1}{8} + \frac{5}{72}\eta_4 - \frac{1}{6}\eta_2)x^3 + (\frac{1}{8} + \frac{1}{24}\eta_3^2 - \frac{1}{24}\eta_4)x$$

$$(3.29) \quad p_3 = -\frac{1}{10368}\eta_3^3x^8 + (\frac{1}{96}\eta_3 + \frac{19}{10368}\eta_3^3 - \frac{1}{72}\eta_3\eta_2 + \frac{5}{864}\eta_4\eta_3)x^6$$

$$\begin{aligned}
 & + \left(\frac{19}{1728}\eta_3^3 - \frac{1}{30}\eta_5 + \frac{1}{8}\eta_6 - \frac{1}{72}\eta_4\eta_3 \right)x^4 + \left(-\frac{5}{96}\eta_4\eta_3 + \frac{35}{864}\eta_3^3 + \frac{1}{32}\eta_3 + \frac{1}{80}\eta_5 \right)x^2 \\
 & + \frac{35}{432}\eta_3^3 - \frac{5}{48}\eta_4\eta_3 + \frac{1}{40}\eta_5 + \frac{1}{16}\eta_3
 \end{aligned}$$

The expansion for the distribution function of the maximum likelihood estimator is now

$$(3.30) \quad G_n(x) = \Phi(x) + \frac{1}{\sqrt{n}}p_1(x)\phi(x) + \frac{1}{n}p_2(x)\phi(x) + \frac{1}{n^{3/2}}p_3(x)\phi(x) + o(n^{-3/2}),$$

with $G_n(x) = P_0(\sqrt{n}\hat{\theta}_n \leq x)$.

3.3. Finding the Cornish-Fisher expansion for $G_n^{-1}(u)$. We will now find a Cornish-Fisher expansion for $G_n^{-1}(\cdot)$.

Assume that G_n^{-1} is of the form

$$(3.31) \quad G_n^{-1}(u) = z_u + \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{3/2}},$$

where z_u denotes $\Phi^{-1}(u)$. We construct Taylor expansions for the terms at the right hand side of (3.30) and plug $G_n^{-1}(u)$ into these expansions.

The first term of (3.30)

$$\begin{aligned}
 (3.32) \quad \Phi(G_n^{-1}(u)) &= \Phi\left(z_u + \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{3/2}}\right) \\
 &= \Phi(z_u) + \left(\frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{3/2}}\right)\phi(z_u) + \frac{1}{2}\left(\frac{A}{\sqrt{n}} + \frac{B}{n}\right)^2 \cdot -z_u\phi(z_u) \\
 &\quad + \frac{1}{6}\left(\frac{A}{\sqrt{n}}\right)^3(z_u^2 - 1)\phi(z_u) \\
 &= u + \left[\frac{A}{\sqrt{n}} + \frac{1}{n}\left(-\frac{1}{2}z_u A^2 + B\right)\right. \\
 &\quad \left.+ \frac{1}{n^{3/2}}\left(-\frac{1}{6}A^3 + \frac{1}{6}A^3 z_u^2 - ABz_u + C\right)\right]\phi(z_u)
 \end{aligned}$$

The second term of (3.30)

$$\begin{aligned}
 (3.33) \quad & \frac{1}{\sqrt{n}}p_1\left(z_u + \frac{A}{\sqrt{n}} + \frac{B}{n}\right)\phi\left(z_u + \frac{A}{\sqrt{n}} + \frac{B}{n}\right) \\
 &= \frac{1}{\sqrt{n}}\left(-\frac{1}{12}\eta_3\left((z_u + \frac{A}{\sqrt{n}} + \frac{B}{n})^2 + 2\right)\right)[1 - z_u\left(\frac{A}{\sqrt{n}} + \frac{B}{n}\right) + \frac{1}{2}(z_u^2 - 1)\left(\frac{A}{\sqrt{n}}\right)^2]\phi(z_u) \\
 &= \left[\frac{1}{\sqrt{n}}\left(-\frac{1}{12}\eta_3 z_u^2 - \frac{1}{6}\eta_3\right) + \frac{z_u^3 A \eta_3}{12n} + \frac{(-\frac{1}{24}z_u^4 A^2 \eta_3 + \frac{1}{8}z_u^2 A^2 \eta_3 + \frac{1}{12}z_u^3 B \eta_3)}{n^{3/2}}\right]\phi(z_u)
 \end{aligned}$$

The third term of (3.30)

$$\begin{aligned}
(3.34) \quad & \frac{1}{n} p_2(z_u + \frac{A}{\sqrt{n}}) \phi(z_u + \frac{A}{\sqrt{n}}) \\
&= \frac{1}{n} \left(-\frac{1}{288} \eta_3^2 (z_u + \frac{A}{\sqrt{n}})^5 + \left(\frac{1}{72} \eta_3^2 + \frac{1}{8} + \frac{5}{72} \eta_4 - \frac{1}{6} \eta_2 \right) (z_u + \frac{A}{\sqrt{n}})^3 \right. \\
&\quad \left. + \left(\frac{1}{8} + \frac{1}{24} \eta_3^2 - \frac{1}{24} \eta_4 \right) (z_u + \frac{A}{\sqrt{n}}) \left[1 - z_u \left(\frac{A}{\sqrt{n}} \right) \right] \phi(z_u) \right) \\
&= \left[\frac{1}{n} \left(-\frac{1}{288} \eta_3^2 z_u^5 + \left(\frac{1}{8} + \frac{1}{72} \eta_3^2 + \frac{5}{72} \eta_4 - \frac{1}{6} \eta_2 \right) z_u^3 + \left(\frac{1}{8} + \frac{1}{24} \eta_3^2 - \frac{1}{24} \eta_4 \right) z_u \right) \right. \\
&\quad \left. + \frac{1}{n^{3/2}} \left(\frac{1}{288} z_u^6 \eta_3^2 + \left(-\frac{5}{72} \eta_4 + \frac{1}{6} \eta_2 - \frac{1}{32} \eta_3^2 - \frac{1}{8} \right) z_u^4 \right. \right. \\
&\quad \left. \left. + \left(\frac{1}{4} + \frac{1}{4} \eta_4 - \frac{1}{2} \eta_2 \right) z_u^2 + \frac{1}{8} + \frac{1}{24} \eta_3^2 - \frac{1}{24} \eta_4 \right) A \right] \phi(z_u)
\end{aligned}$$

The last term of (3.30)

$$\begin{aligned}
(3.35) \quad & \frac{1}{n^{3/2}} p_3(z_u) \phi(z_u) \\
&= \left(-\frac{1}{10368} \eta_3^3 z_u^8 + \left(\frac{1}{96} \eta_3 + \frac{19}{10368} \eta_3^3 - \frac{1}{72} \eta_3 \eta_2 + \frac{5}{864} \eta_4 \eta_3 \right) z_u^6 \right. \\
&\quad \left. + \left(\frac{19}{1728} \eta_3^3 - \frac{1}{30} \eta_5 + \frac{1}{8} \eta_6 - \frac{1}{72} \eta_4 \eta_3 \right) z_u^4 + \left(-\frac{5}{96} \eta_4 \eta_3 + \frac{35}{864} \eta_3^3 + \frac{1}{32} \eta_3 + \frac{1}{80} \eta_5 \right) z_u^2 \right. \\
&\quad \left. + \frac{35}{432} \eta_3^3 - \frac{5}{48} \eta_4 \eta_3 + \frac{1}{40} \eta_5 + \frac{1}{16} \eta_3 \right) \phi(z_u)
\end{aligned}$$

We take all the terms of the order $1/\sqrt{n}$ together to find A .

$$(3.36) \quad A = \frac{\eta_3}{12} (z_u^2 + 2).$$

Next, we take all the terms of order $1/n$, plug in the found A , to get B

$$(3.37) \quad B = \left(-\frac{1}{8} - \frac{1}{72} \eta_3^2 - \frac{5}{72} \eta_4 + \frac{1}{6} \eta_2 \right) z_u^3 + \left(-\frac{1}{36} \eta_3^2 - \frac{1}{8} + \frac{1}{24} \eta_4 \right) z_u$$

By plugging A and B in (3.32), (3.33), (3.34), (3.35) and taking all the $n^{-3/2}$ terms together we find

$$\begin{aligned}
(3.38) \quad C &= \left(-\frac{1}{48} \eta_3 - \frac{1}{144} \eta_4 \eta_3 + \frac{1}{24} \eta_3 \eta_2 + \frac{1}{30} \eta_5 - \frac{1}{8} \eta_6 - \frac{19}{1728} \eta_3^3 \right) z_u^4 \\
&\quad + \left(-\frac{5}{48} \eta_3 + \frac{1}{12} \eta_3 \eta_2 - \frac{1}{80} \eta_5 - \frac{67}{1296} \eta_3^3 + \frac{1}{48} \eta_4 \eta_3 \right) z_u^2 \\
&\quad - \frac{1}{12} \eta_3 - \frac{1}{40} \eta_5 + \frac{1}{9} \eta_4 \eta_3 - \frac{113}{1296} \eta_3^3.
\end{aligned}$$

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